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# VIBRATION ANALYSIS OF PLATES USING THE MULTIVARIABLE SPLINE ELEMENT METHOD

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**Abstract**—The vibration analysis of plates using the multivariable spline element method is presented in this paper. The spline functions are applied to construct bending moments, twisting moments and transverse displacement field functions. The spline equations of eigenvalue problems with multiple variables of vibration of plates are derived based on the Hellinger–Reissner mixed variational principle. For simplicity, the boundary conditions which consist of three local spline points are amended to fit any specified boundary conditions. Several numerical solutions of plate vibration analysis are presented which illustrate the accuracy and convergence of the method.

## INTRODUCTION

The spline finite element method has been developed in the last 10 years. Antes (1974) and Shih (1979) presented bicubic splines as displacement functions for plate bending problems. Mizusawa *et al.* (1979) used the bicubic spline functions to solve the vibration and buckling problems for skew plates. Shen *et al.* (1987), Shen and Wang (1989) and Shen and Huang (1990) extended the spline element method for analysing static, dynamic vibration and stability of stiffened plate and shell problems. The spline element method with a single variable for analysing plates and shells has fewer unknowns, a high degree of accuracy and can be easily programmed, so it was successfully used to analyse certain types of structures with a regular shape on a microcomputer.

Based on the development of the finite element method, the mixed finite element method which is applied to the plate and the plane stress–strain problems is presented in Herrmann (1967) and Mirza and Olson (1980). A kind of multivariable spline element method for analysing plate bending problems has been presented by Shen and Kan (1991). In the present study, bicubic B splines have been used to construct the bending moments, the twisting moments and the transverse displacement field functions in the analysis of the vibration of plates. The spline element vibration eigenvalue equations are derived based on the Hellinger–Reissner mixed variational principle. To demonstrate the accuracy of this method, several numerical examples are given. The proposed multivariable spline element method is compared with other numerical methods. It is shown to be quite effective for the solution of the vibration of plate problems.

## SPLINE INTERPOLATE FUNCTIONS AND FIELD FUNCTIONS

In the multivariable spline finite element method, the bending moments, the twisting moments and the transverse displacement are all chosen as field functions. The field functions of the plate are defined in the form of the Kronecker product of two B cubic spline functions:

$$\{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \lfloor\Phi(x)\rfloor \oplus \lfloor\Phi(y)\rfloor & & \bigcirc \\ & \lfloor\Phi(x)\rfloor \oplus \lfloor\Phi(y)\rfloor & \\ \bigcirc & & \lfloor\Phi(x)\rfloor \oplus \lfloor\Phi(y)\rfloor \end{bmatrix} \begin{Bmatrix} \{A\} \\ \{B\} \\ \{C\} \end{Bmatrix}, \tag{1}$$

$$w = \lfloor\Phi(x)\rfloor \oplus \lfloor\Phi(y)\rfloor\{D\}, \tag{2}$$

where

$$\lfloor\Phi(x)\rfloor = [\phi_{-1}(x)\phi(x)_0(x)_1 \ldots \phi(x)_N\phi(x)_{N+1}], \quad (x = x, y) \tag{3}$$

$$\{A\} = [\bar{a}_{-1}\bar{a}_0\bar{a}_1 \ldots \bar{a}_s \ldots \bar{a}_{N+1}]^{\mathrm{T}}, \tag{4}$$

$$\bar{a}_s = [a_{-1s}a_{0s}a_{1s} \ldots a_{ss} \ldots a_{N+1s}], \quad (s = -1, 0, 1, \ldots, M, M+1), \tag{5}$$

$\oplus$ —— Kronecker product of two matrices, i.e. $[A] \oplus [B] = (a_{ij}[B])$.

The parameter vectors $\{A\}$, $\{B\}$, $\{C\}$ and $\{D\}$ are all constants which are to be determinated. Parameter vectors $\{B\}$, $\{C\}$ and $\{D\}$ are similar to parameter vector $\{A\}$. The basis of the splines can be written as follows:

When $N \geqslant 4$

$$\phi_{-1} = \varphi_3\left(\frac{x}{h}+1\right),$$

$$\phi_0(x) = \varphi_3\left(\frac{x}{h}\right) - 4\varphi_3\left(\frac{x}{h}+1\right),$$

$$\phi_1(x) = \varphi_3\left(\frac{x}{h}-1\right) - \tfrac{1}{2}\varphi_3\left(\frac{x}{h}\right) + \varphi_3\left(\frac{x}{h}+1\right),$$

$$\phi_2(x) = \varphi_3\left(\frac{x}{h}-2\right),$$

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$$\phi_{N-2}(x) = \varphi_3\left(\frac{x}{h}-N+2\right),$$

$$\phi_{N-1}(x) = \varphi_3\left(\frac{x}{h}-N+1\right) - \tfrac{1}{2}\varphi_3\left(\frac{x}{h}-N\right) + \varphi_3\left(\frac{x}{h}-N-1\right),$$

$$\phi_N(x) = \varphi_3\left(\frac{x}{h}-N\right) - 4\varphi_3\left(\frac{x}{h}-N-1\right),$$

$$\phi_{N+1}(x) = \varphi_3\left(\frac{x}{h}-N-1\right). \tag{6}$$

The figures of splines $\varphi_3(x)$, $\varphi_3((x/h)-i)$, $\phi_i(x)$, $(i = -1, 0, 1, \ldots, N+1)$ are shown in Figs 1, 2 and 3 where,

$$\varphi_3(x) = \frac{1}{6h^3}\begin{cases} (x-x_{i-2})^3 & x_{i-2} \leqslant x \leqslant x_{i-1} \\ h^3+3h^2(x-x_{i-1})+3h(x-x_{i-1})^2-3(x-x_{i-1})^3 & x_{i-1} \leqslant x \leqslant x_i \\ h^3+3h^2(x_{i+1}-x)+3h(x_{i+1}-x)^2-3(x_{i+1}-x)^3 & x_i \leqslant x \leqslant x_{i+1} \\ (x_{i+2}-x)^3 & x_{i+1} \leqslant x \leqslant x_{i+2} \\ 0 & x_{i+2} < |x| \end{cases} \tag{7}$$

In order to be simplified for the treatment of boundary conditions, for $N \geqslant 4$, the linear combination of splines has been constructed for the first three splines and the last three splines which are shown in Fig. 3.

At the left-hand point, $x = 0$, we have

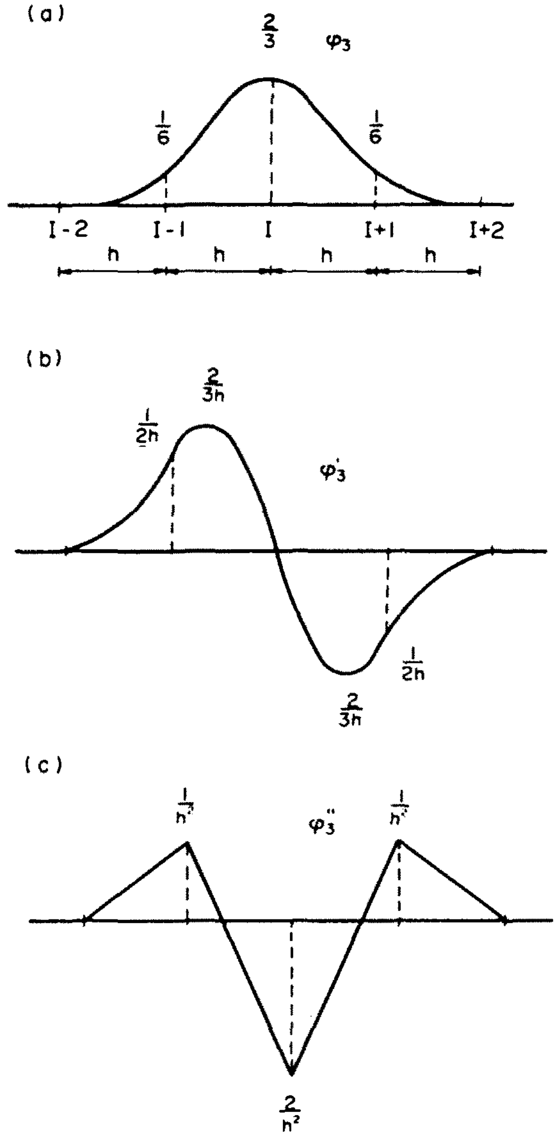


Fig. 1.

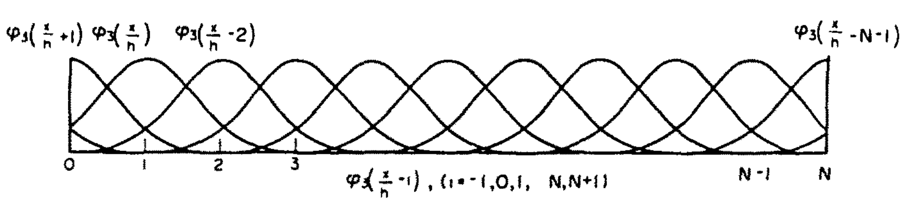


Fig. 2.

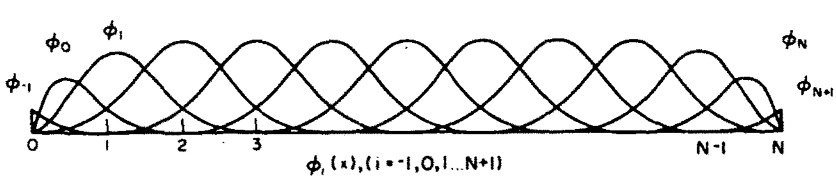


Fig. 3.

$$\phi_i(0) = 0, \quad (i \neq -1),$$
$$\phi_i'(0) = 0, \quad (i \neq -1, 0).$$

At the right-hand point, $x = a$, we have

$$\phi_i(a) = 0, \quad (i \neq N+1),$$
$$\phi_i'(a) = 0, \quad (i \neq N, N+1).$$

## MIXED VARIATIONAL PRINCIPLE OF THIN PLATE

The mixed variational principle of thin plates is given by Hu (1981):

$$\Pi_2 = \int\int_\Omega \{M\}^T\{\chi\}\,dx\,dy - \int\int_\Omega \tfrac{1}{2}\{M\}^T[D]^{-1}\{M\}\,dx\,dy - \int\int_\Omega qw\,dx\,dy$$
$$-\int_{C_1+C_2}\left(\frac{\partial M_{ns}}{\partial s}+Qn\right)(w-\bar{w})\,ds - \int_{c_3}\bar{q}w\,ds + \int_{c_1} Mn\left(\frac{\partial w}{\partial n}-\bar{\varphi}\right)ds + \int_{c_2+c_3} Mn\frac{\partial w}{\partial n}\,ds, \tag{8}$$

$c_1$—on fixed edges, $w = \bar{w}, \dfrac{\partial w}{\partial n} = \bar{\varphi}$,

$c_2$—on simply-supported edges, $w = \bar{w}, Mn = \bar{M}n$,

$c_3$—on free edges, $Mn = \bar{M}n, \dfrac{\partial Mns}{\partial s} + Qn = \bar{q}$.

In the case of homogeneous boundary conditions, the mixed functional for vibration problems becomes as follows:

$$\Pi_2 = \int\int_\Omega \{M\}^T\{\chi\}\,dx\,dy - \int\int_\Omega \tfrac{1}{2}\{M\}^T[D]^{-1}\{M\}\,dx\,dy - \int\int_\Omega \tfrac{1}{2}\lambda\bar{m}w^2\,dx\,dy, \tag{9}$$

where

$$\{M\} = [M_x \quad M_y \quad M_{xy}]^T, \tag{10}$$

$$\{\chi\} = [-w_{,xx} - w_{,yy} - 2w_{,x,y}]^T \tag{11}$$

$$[D] = \frac{Et^3}{12(1-\mu^2)}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}, \tag{12}$$

$\bar{m}$—mass density of plate, $\lambda$—vibration eigenvalue of plate. If we take the moments and displacements as variations to the mixed functional and make the variation of $\Pi_2$ equal to zero, i.e. $\delta\Pi_2 = 0$, we can obtain the equations including the equations of equilibrium, boundary and geometry of the plate.

## MULTIVARIABLE SPLINE VIBRATION EIGENMODE EQUATION OF PLATE

The bending moments, twisting moments and transverse displacements used as field variables and the bicubic $B_3$ spline functions are used to construct the field functions with multiple variables. Taking eqns (1) and (2) and substituting them into eqn (9) and applying the mixed energy principle, we have

$$\frac{\partial \Pi_2}{\partial\{A\}} = \{0\},\quad \frac{\partial \Pi_2}{\partial\{B\}} = \{0\},\quad \frac{\partial \Pi_2}{\partial\{C\}} = \{0\},\quad \frac{\partial \Pi_2}{\partial\{D\}} = \{0\}, \tag{13}$$

which yields

$$\left[\begin{array}{c|c} [F] & [H] \\ \hline [H]^{\mathrm{T}} & [0] \end{array}\right] \begin{Bmatrix} \{A\} \\ \{B\} \\ \{C\} \\ \hline \{D\} \end{Bmatrix} = \left[\begin{array}{c|c} [0] & [0] \\ \hline [0] & \lambda[M] \end{array}\right] \begin{Bmatrix} \{A\} \\ \{B\} \\ \{C\} \\ \hline \{D\} \end{Bmatrix}, \tag{14}$$

where

$$\begin{bmatrix} -\frac{12}{Et^3}[K_x^{00}]\otimes[K_y^{00}] & \frac{12\mu}{Et^3}[K_x^{00}]\otimes[K_y^{00}] & 0 \\ \text{Symm.} & -\frac{12}{Et^3}[K_x^{00}]\otimes[K_y^{00}] & 0 \\ & & -\frac{24(1+\mu)}{Et^3}[K_x^{00}]\otimes[K_y^{00}] \end{bmatrix}, \tag{15}$$

$$[H] = \begin{bmatrix} -[K_x^{22}]\otimes[K_y^{00}] \\ -[K_x^{00}]\otimes[K_y^{22}] \\ -2[K_x^{01}]\otimes[K_y^{10}] \end{bmatrix}, \tag{16}$$

$$[M] = \bar{m}[K_x^{00}]\otimes[K_y^{00}], \tag{17}$$

where

$$[K_x^{ij}] = \int_0^a [\Phi^i(x)]^{\mathrm{T}}[\Phi^j(x)]\,\mathrm{d}x,\quad (x = x, y), (i, j = 0, 1, 2). \tag{18}$$

The matrices $[K_x^{ij}]$ $(x = x, y;\ i, j = 0, 1, 2)$ can be found in Shen and Wang (1986). From eqn (14) we have the eigenvalue mode vector equation of the vibration analysis of plates in the following form:

$$([H]^{\mathrm{T}}[F]^{-1}[H] - \lambda[M])\{D\} = \{0\}. \tag{19}$$

In eqn (19), the kinematic boundary conditions are easily treated just like the finite element method, but they do not relate to the displacements, only to the spline node parameters.

## NUMERICAL EXAMPLES

Generally, by using refined meshes of spline knots and higher order splines, we can obtain better approximate solutions using the multivariable spline element method. However, the spline functions with more knots and higher order splines can accelerate the convergence for numerical results, but more variables are involved which leads to excessive CPU time, so that it is not suitable for practical computations. Based on our experience, using the bicubic splines and meshes of 4 × 4, 6 × 6, 8 × 8, 10 × 10 will provide enough precision in numerical calculations.

Numerical examples were performed on a VAX-11/780 minicomputer. The proposed method can be used to solve the vibration eigenvalue problems of plates. In the first example,

Table 1. Frequencies of simply-supported plate

| m, n | Meshes | M.S.E.M. | Warburton (1954) |
|---|---|---|---|
| $\omega_{1,1}$ | 8 × 8 | 19.73381 | 19.74 |
| $\omega_{2,1}$ | 8 × 8 | 49.34739 | 49.35 |
| $\omega_{3,1}$ | 8 × 8 | 78.95469 | 78.95 |
| $\omega_{2,2}$ | 8 × 8 | 98.70766 | 98.64 |
| | | $\alpha^*$ | |

$\omega = \alpha^*(1/L^2)(D/\bar{m}t)^{1/2}$.
$\mu = 0.3$.
M.S.E.M.—Multivariable Spline Element Method.

Table 2. Frequencies of plate with simply-supported edges for different meshes

| Meshes | | | | |
|---|---|---|---|---|
| 4 × 4 | 19.73923 | 49.38513 | 79.01062 | 100.75174 |
| 6 × 6 | 19.73886 | 49.34992 | 78.95966 | 98.79032 |
| 8 × 8 | 19.73381 | 49.34739 | 78.95649 | 98.70766 |
| * | 19.74 | 49.35 | 78.95 | 98.64 |

* Warburton (1954).

Table 3. Frequencies of plate with fixed edges for different meshes

| Meshes | | | | |
|---|---|---|---|---|
| 4 × 4 | 36.01731 | 73.74091 | 108.82559 | 134.59386 |
| 6 × 6 | 35.99366 | 73.43199 | 108.32896 | 132.00426 |
| 8 × 8 | 35.98793 | 73.40506 | 108.25762 | 131.64781 |
| * | 35.99 | 73.41 | 108.3 | 131.6 |

* Blevins (1979).

Table 4. Frequencies of plate with two opposite edges simply supported and other edges fixed for different meshes

| Meshes | | | | | | |
|---|---|---|---|---|---|---|
| 4 × 4 | 28.95979 | 54.86541 | 69.59309 | 94.90040 | 104.38136 | 131.90736 |
| 6 × 6 | 28.95244 | 54.76415 | 69.33943 | 94.62939 | 102.36386 | 129.46341 |
| 8 × 8 | 28.95030 | 54.74785 | 69.32851 | 94.59936 | 102.24642 | 129.13942 |
| * | 28.95 | 54.74 | 69.32 | 94.59 | 102.2 | |

* Blevins (1979).

Table 5. Frequencies of plate with one edge fixed and other three edges simply supported for different meshes

| Present method | Meshes | | | | | |
|---|---|---|---|---|---|---|
| | 4 × 4 | 23.65015 | 51.74423 | 58.74947 | 86.28145 | 102.36620 |
| | 6 × 6 | 23.64658 | 51.68347 | 58.65156 | 85.15998 | 100.38531 |
| | 8 × 8 | 23.64513 | 51.67347 | 58.64604 | 86.14058 | 100.28543 |
| | * | 23.64 | 51.67 | 58.65 | 86.13 | 100.3 |

* Blevins (1979).

simply supported with four edges is a square plate; the second example is also a square plate with four fixed edges; the third and fourth examples are square plates with two opposite edges simply supported and the other edges fixed.

The numerical results of the vibration analysis for the plates are shown in the tables. In Table 1, the vibration frequencies are shown for a square plate with simply-supported edges. In Table 2, the vibration frequencies for the meshes are shown. The results rapidly converge to the exact solution. The lowest frequencies of different supported plates are shown in Table 6.

Table 6. Lowest frequencies of thin plate

| Plate | Meshes | M.S.E.M. | Warburton (1954) |
|---|---|---|---|
| SSSS | 4 × 4 | 19.73923 | 19.74 |
| | 6 × 6 | 19.73886 | |
| | 8 × 8 | 19.73381 | |
| FFFF | 4 × 4 | 36.01731 | 35.99 |
| | 6 × 6 | 35.99366 | |
| | 8 × 8 | 35.98793 | |
| SFSF | 4 × 4 | 28.95979 | 28.95 |
| | 6 × 6 | 28.95244 | |
| | 8 × 8 | 28.95030 | |
| SSSF | 4 × 4 | 23.65015 | 23.65† |
| | 6 × 6 | 23.64658 | |
| | 8 × 8 | 23.64513 | |

S—Simply-supported edge.
F—Fixed edge.
† R. D. Blevins (1979).

## CONCLUSIONS

Bicubic spline functions are used to construct moments, twisting moments and displacements as field variables for the analysis of plate vibrations in this paper. Multivariate spline eigenvalue equations are derived based on the Hellinger–Reissner mixed energy principle. The present results demonstrate the good convergence characteristics of the multivariable spline element method. The spline functions also have the desired properties associated with piecewise polynomials so that the present method has a high degree of accuracy for the vibration analysis of plates.

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## REFERENCES

Antes, H. (1974). Bicubic fundamental splines in plate bending. *Int. J. Num. Meth. Engng* **8**, 503–511.
Blevins, R. D. (1979). *Formulas for Natural Frequency and Mode Shape*. Van Nostrand, Reinhold Co.
Herrmann, L. R. (1967). Finite element bending analysis for plates. *Proc. ASCE., Engng Mech. Div.* **93**, 13–26.
Hu, HaiChang (1981). *Variational Principles and Applications in Theory of Elasticity*. Science Press, Beijing, China.
Mirza, F. A. and Olson, M. D. (1980). The mixed finite element method in plane elasticity. *Int. J. Numer. Meth. Engng* **15**, 273–289.
Mizusawa, T., Kajita, T. and Naruoka, M. (1979). Vibration of skew plates using B spline functions. *J. Sound Vibr.* **62**(2), 301–308.
Shen PengCheng and Huang DaDe (1990). Dynamic analysis of stiffened plates and shells using spline Gauss collocation method. *Comput. Struct.* **36**(4), 623–629.
Shen PengCheng, Huang DaDe and Wang ZonGmon (1987). Static, vibration and stability analysis of stiffened plates using spline functions. *Comput. Struct.* **27**(1), 73–78.
Shen PengCheng and Wang JianGuo (1986). Solution of governing differential equations of vibrating cylindrical shells using B spline functions. *Int. J. Numer. Meth. for Partial Diff. Eqns* **2**, 173–185.
Shen PengCheng and Wang JianGuo (1989). A semianalytical method for static analysis of shallow shells. *Comput. Struct.* **31**(5), 825–831.
Shen PengCheng and Kan HongBo (1991). Multivariable spline element analysis for plate bending problems. *Comput. Struct.* **40**(6), 1343–1349.
Shih ChungTze (1979). On spline finite element method. *Math. Numer. Sinic.* **1**, 50–72 (in Chinese).
Warburton, G. B. (1954). The vibration of rectangular plates. *Proc. Inst. Mech. Engrs* **68**(12), 371–384.